International Journal on
"Technical and Physical Problems of Engineering"
(IJTPE)
Published by International Organization of IOTPE

ISSN 2077-3528
IJTPE Journal
www.iotpe.com
ijtpe@iotpe.com

December 2013 | Issue 17 | Volume 5 | Number 4 | Pages 153-160

INVESTIGATING ARC LENGTH METHOD TO GET FAST CONVERGENCE IN 2D TRUSS STRUCTURS

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Abstract- The stability analysis of slender structures requires carrying out geometrically nonlinear analysis. By following the nonlinear equilibrium path, it is possible to understand the phenomenon of collapse or buckling or the total bearing capacity of structures. Nonlinear equilibrium equations in the analysis of structures are often solved by using Newton-Raphson method, which is an incremental iterative procedure. However, the method diverges when reaches to a limit point. Therefore, only a part of the curve is obtained. To overcome the difficulties with limit points, displacement control techniques were introduced. The arc-length method is among the methods that were developed as an effort to enable solution algorithms to pass critical points. In this research, influence of incremental length size (Δl) for the arc-length control method was studied and implemented using Matlab software. To get fast convergence of the arc-length method, the increasing of incremental length causes convergence to be accelerated. However, the accuracy is decreased. To overcome this problem, a new constraint equation is suggested so that it helps the increasing of the accuracy.

Keywords: Arc-Length Method, Nonlinear Analysis, Limit Points, Load-Displacement Path.

I. INTRODUCTION

Some of structures behave in a linear elastic fashion under service loads. The premise of linear elastic behavior forecloses the possibility of revealing any manifestation of nonlinearity. In this case, crucial information may be missing. However, prior to reaching the limit of resistance, almost all structures would exhibit significant nonlinear response. By using nonlinear analysis, the uncertainty regarding actual behavior may be reduced. In the process, however, the element of art in modeling the structure and in handling the equations of analysis is increased.

In modeling, the analyst must decide what sources of nonlinearity are appropriate to be significant and how to represent them [1]. The stability analysis of slender structures requires carrying out geometrically nonlinear analysis [2]. With nonlinear structural analysis, buckling loads can be determined by checking the actual structural deformations during loading. The buckling strength of the structure and some other factors can be achieved. To get all these goals, it is necessary to draw full equilibrium path.

This purpose requires carrying out nonlinear analysis to find out load-displacement diagram. In the early 60's, the incremental technique in tracing the load-displacement curve of trusses and frames was developed by Turner et al [3] and Argyris [4]. However, the technique diverges when reaches to a limit point. Therefore, only a part of the curve is obtained. To overcome the difficulties with limit points, Zienkiewicz [5] proposed the displacement method.

In this approach, a component of displacement vector remains constant. Batoz and Dhatt [6] proposed a simpler procedure of introducing the constant displacement constraint, which retains the symmetrical character of the tangent stiffness matrix and hence their method is convenient and efficient for computer programming. The constant displacement method is only capable of handling snap-through but not snap-back problems. To overcome these problems, the arc-length method was proposed. The arc-length method was then developed as another effort to enable solution algorithms to pass critical points.

In relation to structural analysis, Riks [7] and Wempner [8] published the first attempt in this respect, using a linear constraint equation such that the iterative change was normal to the tangent. Later, several scholars modified the method by the means of altering the constraint equation, and therefore, the way of the corrector steps of the iterative procedure was developed. For example, Ramm [9] used a different linear constraint such that iterative change was normal to the secant change. The previous two methods were the first versions of the linearized arc-length method.

At the same time, Crisfield did a further modification, introducing the spherical arc-length method [10], which uses a quadratic constraint or the Euclidean norm of the incremental displacement to a fixed quantity. To avoid the problems that arise in choice of a proper root, as Crisfield method required, a consistently linearized version of arc-length method using the same quadratic constraint was proposed by Schweizerhof and Wriggers [11].

II. ARC-LENGTH CONTROL METHOD

The Load and Displacement Control methods fail to draw the complete equilibrium path in the presence of limit and turning points. The arc-length control method is intended to handle these critical points and draw the entire load-displacement response diagram.

The starting point of the arc-length control is the equilibrium equation in the form of the residuum r.

$$r(d,\lambda) = q_i(d) - \lambda q_e = 0 \quad (1)$$

where, q_i is the internal force vector of the structure which is the function of the nodal displacement d, the vector q_e is the external load vector and the scalar λ is again the load level parameter. Further to the equilibrium Equation (1), an additional constraint is added to complete the set of equations. This constraint equation is the arc length s, defined by:

$$s = \int ds \quad (2)$$

$$ds = \sqrt{dd^T.dd + d\lambda^2.\psi^2.q_e^T.q_e} \quad (3)$$

The arc-length method is then aimed to find the intersection of a given arc length s with the equilibrium equation, such that:

$$r(s) = q_i(d(s)) - \lambda(s)q_e = 0 \quad (4)$$

To solve Equation (4), a predictor-corrector scheme is used. Initially, the differential form of Equation (3) can be replaced with an incremental form, such that:

$$a^i = \left(\Delta d^i\right)^T \Delta d^i + \left(\Delta\lambda^i\right)^2 \psi^2 q_e^T q_e - \Delta l^2 = 0 \quad (5)$$

where, Δl is the fixed radius of the desired intersection with the equilibrium path, also known as incremental length (Figure 1). In the same figure, the vector Δd and the scalar $\Delta\lambda$ are in incremental form and are related back to the last converged equilibrium state. Meanwhile, δd and $\delta\lambda$ are the iterative displacement vector and load level respectively. These are associated to the previous iteration step that, in most cases, is not an equilibrium state.

In addition, the scaling parameter ψ is required in Equation (5) to combine different dimensions for load and displacement terms. The main essence of the arc-length method is that the load level λ becomes now a parameter. Therefore, the total unknowns are n+1, n from the displacement variables of vector d and one from the load parameter λ. To solve this, Equation (1) gives a total of n equilibrium equations, while Equation (5) gives one constraint equation. These n+1 equations can be solved iteratively by applying the Newton-Raphson method to Equation (1) and Equation (5). Riks [7] and Wempner [8] first proposed this idea, though with a different constraint equation. A truncated Taylor series of Equation (1) and Equation (5) respectively yields:

$$r^{i+1} = r^i + \frac{\partial r}{\partial d}\delta d^i + \frac{\partial r}{\partial \lambda}\delta\lambda^i = r^i + K_t\delta d^i - q_e\delta\lambda^i = 0 \quad (6)$$

$$\begin{cases} a^{i+1} = a^i + \dfrac{\partial a}{\partial d}\delta d^i + \dfrac{\partial a}{\partial \lambda}\delta\lambda^i \Rightarrow \\ a^{i+1} = a^i + 2\left(\Delta d^i\right)^T \delta d^i + 2\Delta\lambda^i\delta\lambda^i\psi^2 q_e^T q_e = 0 \end{cases} \quad (7)$$

where, K_t is the tangent stiffness matrix. The other terms have been already defined. Equation (6) and Equation (7) can be combined in a different way.

$$\begin{pmatrix} \delta d^i \\ \delta\lambda^i \end{pmatrix} = -\begin{bmatrix} K_t & -q_e \\ 2\left(\Delta d^i\right)^T & 2\Delta\lambda^i\delta\lambda^i\psi^2 q_e^T q_e \end{bmatrix}^{-1} \begin{pmatrix} r^i \\ a^i \end{pmatrix} \quad (8)$$

At this point, there are different ways to obtain the solution of Equation (8). Two different possibilities are given in follows.

III. LINEARIZED ARC-LENGTH METHOD

The simplest way is, to solve Equation (8) directly [11]. One may first find an expression for δd and then replace it into $\delta\lambda$. Consequently, from Equation (6), δd becomes:

$$\delta d^i = -K_t^{-1}r^i + \delta\lambda^i K_t^{-1}q_e \quad (9)$$

Therefore by means of the two iterative displacement vectors $\delta\bar{d}$ and δd_t, Equation (9) can be re-expressed as:

$$\delta d^i = \delta\bar{d}^i + \delta\lambda^i\delta d_t^i \quad (10)$$

$$\begin{cases} \delta\bar{d}^i = -K_t^{-1}r^i \\ \delta d_t^i = K_t^{-1}q_e \end{cases} \quad (11)$$

By inserting of Equation (10) into Equation (7) yields:

$$2\left(\Delta d^i\right)^T\left(\delta\bar{d}^i + \delta\lambda^i.\delta d_t^i\right) + 2\Delta\lambda^i.\delta\lambda^i\psi^2 q_e^T q_e = -\frac{a^i}{2} \quad (12)$$

This in turn, by solving for iterative load level $\delta\lambda$ yields:

$$\delta\lambda^i = \frac{-\left(a^i/2\right) - \left(\Delta d^i\right)^T\delta\bar{d}^i}{\left(\Delta d^i\right)^T\delta d_t^i + \Delta\lambda^i.\psi^2 q_e^T q_e} \quad (13)$$

After obtaining δd and $\delta\lambda$, new incremental displacements and load level are:

$$\Delta d^{i+1} = \Delta d^i + \delta d^i \quad (14)$$

$$\Delta\lambda^{i+1} = \Delta\lambda^i + \delta\lambda^i \quad (15)$$

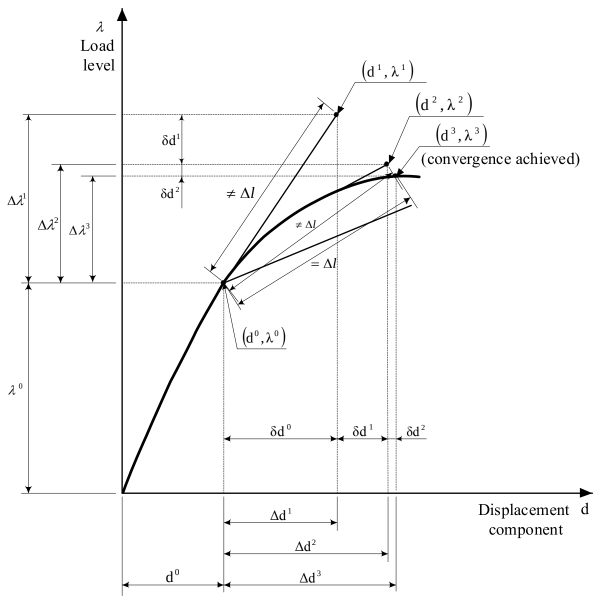

Figure 1. The linearized arc-length method [2]

IV. THE PREDICTOR SOLUTION

The predictor step will indicate the direction of the first attempt to find the equilibrium path, and therefore its importance in the whole iterative process. The most popular idea is to use the forward-Euler scheme to obtain an expression for predictor. According to Figure 2 we have:

$$\Delta d^1 = K_t^{-1}\Delta q_e \quad (16)$$

where, K_t is the tangent stiffness matrix at the beginning of the increment. The superscript of the incremental displacement vector Δd stands for the number of the iteration of the predictor step. The incremental external force Δq_e of Equation (16) can be expressed as:

$$\Delta q_e = \Delta\lambda^1 q_e \tag{17}$$

Inserting Equation (17) into Equation (16) yields:

$$\Delta d^1 = \Delta\lambda^1 K_t^{-1} q_e \tag{18}$$

Recalling definition of δd_t the Equation (11) and Equation (18) can be re-expressed as:

$$\Delta d^1 = \Delta\lambda^1 . \delta d_t^0 \tag{19}$$

The previous equation has to be constrained by the incremental length Δl. Hence substituting Equation (19) into Equation (5), and solving for $\Delta\lambda$, finally yields:

$$\Delta\lambda^1 = \pm \frac{\Delta l}{\sqrt{\left(\delta d_t^0\right)^T \delta d_t^0 + \psi^2 q_e^T q_e}} \tag{20}$$

Because of the plus and minus sign in Equation (20), two different predictors are possible. There are several criteria to predict the continuation direction of the equilibrium path, i.e., to decide the sign of $\Delta\lambda^1$. This may seem a trivial choice, but lays in the center of the method and an erroneous selection would lead to unwanted results, such as example, doubling back on the equilibrium path. The three most popular alternatives to determine if $\Delta\lambda^1$ is positive or negative are given in the next sections.

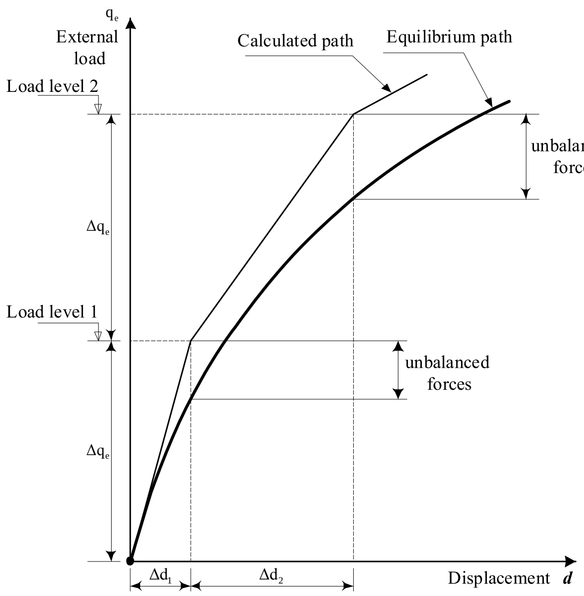

Figure 2. Forward-Euler incremental procedure [2]

V. THE SIGN OF PREDICTOR SOLUTIN

There are three methods to determine the sign of predictor.

A. To Follow the Sign of det(K_t)

$$\operatorname{sgn}\left(\Delta\lambda^1\right) = \operatorname{sgn}\left(\det\left(K_t\right)\right) \tag{21}$$

where, K_t is the tangent stiffness matrix at the beginning of the increment, i.e. when $\Delta\lambda^1$ is computed.

This is the most popular and widely used criterion, which was originally introduced by Crisfield in 1980 [10], and works well in the presence of limit points. However, in the presence of bifurcations, it fails in most cases.

As it will be seen in the next section, through a simple example, the sign of the det(K_t) oscillates from negative to positive around the bifurcation point, and therefore is unable to follow the equilibrium path [12, 13]. This criterion has been shown with pr = 1 in section of numerical examples.

B. To Follow the Sign of the Predictor Work Increment

$$\operatorname{sgn}\left(\Delta\lambda^1\right) = \operatorname{sgn}\left(\left(\delta d_t^0\right)^T q_e\right) \tag{22}$$

where, δd_t^0 is the current tangential solution defined in Equation (11) at the beginning of the incremental and q_e is the external load vector. According to Souza and Feng [14], this criterion is insensitive to bifurcations and can continue to trace an equilibrium path after passing a bifurcation point.

However, this criterion proves ineffective in the descending branch of load-deflection curve in 'snap-back' problems, where the predicted positive 'slope' will provoke a 'back tracing' load increase. In other words, the scheme doubles back on its track after passing a turning point. Some authors have proposed switching between the two previous criteria to overcome their ill behavior at the presence of limit and turning points. However, this represents an extra computational cost. This criteria has been shown with pr = 2 in section of numerical examples.

C. To Follow the Displacements Internal Product

$$\operatorname{sgn}\left(\Delta\lambda^1\right) = \operatorname{sgn}\left(\left(\Delta d^0\right)^T . \delta d_t^0\right) \tag{23}$$

where, Δd^0 is the previous converged incremental displacement and δd_t^0, the current tangential solution defined in Equation (11).

Introduced by Feng et al in 1996 [15], this criterion is insensitive to limit points, turning points and bifurcation points. Its key point is the fact that Δd^0 carries with it information about the history of the current equilibrium path. Its only limitation is the need for a sufficiently small value than Δd^0. However, Feng has claimed that this limitation is already imposed by the Newton-Raphson algorithm and in practice, the required size of Δd^0 for an accurate direction prediction would always be greater than the necessary convergence radius of the N-R scheme.

Nevertheless, in Feng's work, there is no indication for the maximum size of Δd^0 required for the criterion to work properly. Furthermore, there is no information of the very first incremental Δd^0 and therefore the sign of the very first predictor step cannot be determined by Equation (23). A solution would involve the use of the det(K_t) principle and then switch to the proposed criterion. However, it is expected that if the early structure is in stable configuration, the very first predictor load level $\Delta\lambda^1$ will be positive. This criteria has been shown with pr = 3 in section of numerical examples.

VI. THE SCALING PARAMETTER (ψ)

The constraint Equation (5), the base of the arc-length method, reads:

$$a^i = \left(\Delta d^i\right)^T \Delta d^i + \left(\Delta\lambda^i\right)^2 \psi^2 q_e^T q_e - \Delta l^2 = 0 \qquad (24)$$

where, ψ determines influence of load and displacement control in the behavior of the arc-length method. So, for example if $\psi \rightarrow 0$, displacement control likes conduct and if $\psi \rightarrow \infty$, load control likes perform.

Moreover, it can be said [16] load and displacement control methods are particular cases of the arc-length method (Figure 3).

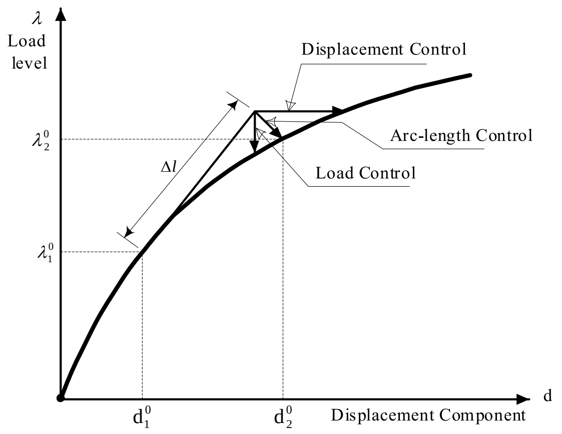

Figure 3. Different path following techniques [2]

The main purpose of ψ in Equation (24) is to give scale between the load and displacement terms, when for example, using great force values for small displacement resultants. The last is very likely to happen in analysis using real values, when the stiffness of the structure elements are high and therefore, large forces are required to obtain small displacements (e.g. $1e^{10}$ Newton to get 1mm displacement). In this case, if $\psi = 1$, the force term $\Delta\lambda^2\psi^2 q_e^T q_e$ in the constraint Equation (24) would tend to infinite if compared to the displacement term $\Delta d^T \Delta d$.

Therefore, the arc-length method would behave similarly to the load control and loosing ability to overcome limit points. However, in practice, it seems that the displacement term, when $\Delta\lambda^2\psi^2 q_e^T q_e \rightarrow 0$, has little effect, and the arc-length method still is capable to pass the bifurcation, limit, and turning points. As a result many authors [9, 13, 17] have advocated the use of $\psi = 0$. Authors like Al-Rasby [18], and Schweiyerhof and Wriggers [11] have already addressed the issue of defining a proper scaling between displacement and load components.

VII. CONVERGENCE CRITERION

In an effective incremental-iterative method, some criteria should be pre-determined for termination or continuation of iterations. If a tight tolerance is selected, excessive computation, effort is spent on unnecessary accuracy. On the other hand, if the tolerance is set too loose, equilibrium error may be excessive and inaccurate solutions can be resulted.

Further to this, the question of whether the equilibrium tolerance should be set on the unbalanced forces or displacements is debatable matter. Through a number of nonlinear analyses by the authors, it was found that a slightly loose tolerance imposed on both the displacement and force error is preferable to a tight tolerance for either the displacement or the force error norm.

For this purpose, 0.1% equilibrium error is allowed for each of the maximum unbalanced displacement and force norms. Equilibrium is only assumed when both of the equilibrium checks are satisfied. Mathematically, the convergence criteria for force and displacement are expressed respectively as:

$$\frac{\sqrt{r \times d}}{\sqrt{f \times d}} < \varepsilon \qquad (25)$$

where, r, f, d are the accumulated residuum force, external load and displacement vectors respectively, ε is the tolerance for equilibrium condition and is set to 0.1% for the present study.

VIII. NUMERICAL EXAMPLES

The numerical examples were chosen to allow comparison between the following purposes:

1- The criteria included in section V to determine the sign of the predictor step in the arc-length method

2- The importance influences of increasing of incremental length (Δl) to get fast convergence

3- Comparison between new and previous constraint equations to get accuracy for the arc-length control method

4- Investigated numerically the scaling parameter ψ

A. Example 1 - The Basic Two Bar Asymmetric System

The classical problem of Figure 4 with two truss elements was analyzed. This example has two limit points on equilibrium path. Figure 5 shows load-displacement curve obtained via the implemented linearized arc-length method with different incremental length (Δl) and criteria to determine the sign of the predictor step. The comparison between curves is obtained with criteria in section V show that all three of criteria have ability to overcome limit point. In other effort to investigate effect of changing value of initial, minimum and maximum incremental length (Δl) to draw load-disp. response diagram in arc-length method, has been shown in Table 1.

The results in Table 1 shows that increase of incremental length (Δl) causes number of points to draw curve can be reduced. According to Table 2, the accuracy of convergence is reduced. To overcome this problem, a new constraint equation is suggested. The Table 2 can be considered to compare the previous and new constraint Equation (5) and Equation (26). As it can be seen from the results of the Table 2, the accuracy of convergence with the new constraint equation is increased. It is noticeable that the values of Table 2 are used for the values of the last row in Table 1.

$$a^i = \left(\left(\Delta d^i\right)^T \Delta d^i + \left(\Delta\lambda^i\right)^2 \psi^2 q_e^T q_e - \Delta l^2\right)^2 = 0 \qquad (26)$$

The choice of a random value for the scaling parameter ψ can lead to the linearized arc-length method to yield disappointing results. If the scaling parameter ψ is set to 1, the force term $\Delta\lambda^2\psi^2 q_e^T q_e$ in the constraint Equation (5) tends to infinite if compared to the displacement term $\Delta d^T \Delta d$.

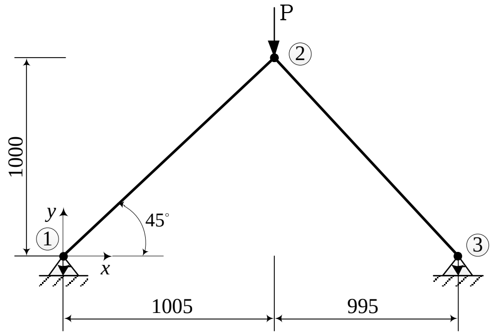

Figure 4. Two bar asymmetric structure

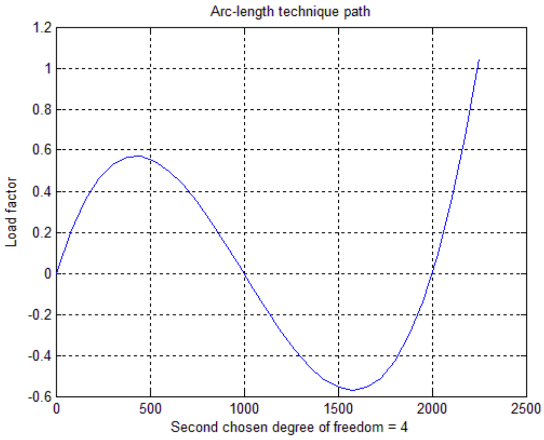

Figure 5. Load-displacement response diagram with Equation (5)

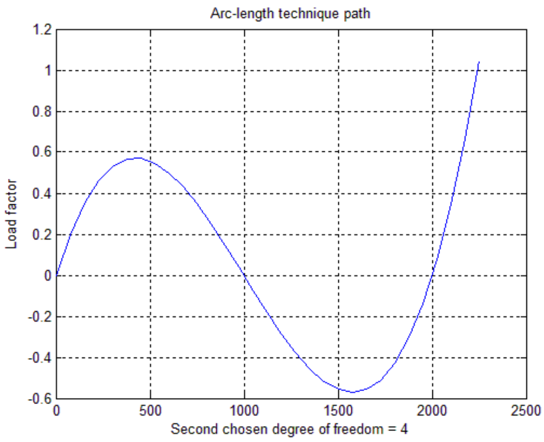

Figure 6. Load-displacement response diagram with Equation (26)

Table 1. Comparison the effect of changing incremental length (Δl)

Initial incremental length	Minimum incremental length	Maximum incremental length	The total number of points on the path
10	5	20	114
20	10	40	58
30	15	60	39
40	20	80	30

Table 2. Comparison the effect of constraint Equations (5) and (26)

Point on the path	Number of iteration	constraint Equation (5)	Constraint Equation (26)
1	1	2.2737×10^{-13}	5.1699×10^{-26}
2	1	-1.3642×10^{-12}	1.8612×10^{-24}
3	1	4.5475×10^{-13}	2.0680×10^{-25}
⋮	⋮	⋮	⋮
28	1	0	0
29	1	-1.8190×10^{-12}	3.3087×10^{-24}
30	1	0	0

Furthermore, the length of the predictor step is reduced and the more number of increments would be needed to draw the same curve. In other words, the arc-length method starts to show a load control-like behavior. On the other hand, if the scaling parameter ψ is set to 0, the size of the predictor step is kept in reasonable values and the lower increments than previous are needed to draw the entire equilibrium path. Additionally, a displacement control-like behavior can be observed from the constant displacement increment size.

However, at least in the analyzed truss structures, the arc-length control method with $\psi = 0$ is capable to overcome all the special points, but only if the criterion for determining the sign of the predictor step yields the right direction.

B. Example 2 - The Basic Two Bar Symmetric System

In this example, the basic two bars symmetric system with the angle between the trusses and the horizontal line is increased to 68° (Figure 7). This system has two bifurcation and two limit points that appear in the equilibrium path. In the Figures 8, 9 and 10, it can be observed how the three studied criteria are affected by the presence of the limit points. The criteria 2 and 3 are not influenced by the limit points and therefore the arc-length method continues drawing the entire primary path.

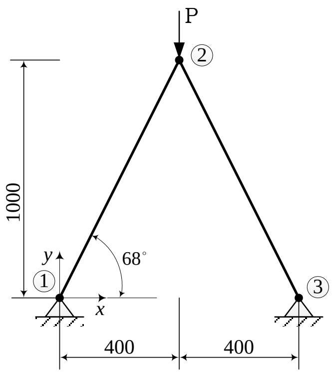

Figure 7. Two bar symmetric structure with augmented angle [2]

The arc-length method, using the criterion 1, fails to overcome the bifurcation point. The problem consist in that when the sign of det(K_t) changes to negative, the predictor step points downwards, and the iterations converge in the equilibrium path that has already been drawn. Therefore, at least in this case, the arc-length method oscillates around a bifurcation point until the maximum number of increments is completed.

$$\partial d^i = -K_t^{-1} r^i \tag{27}$$

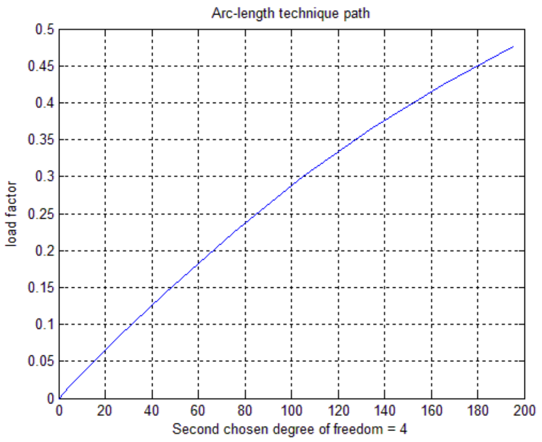

Figure 8. Load-displacement response diagram with *pr* = 1

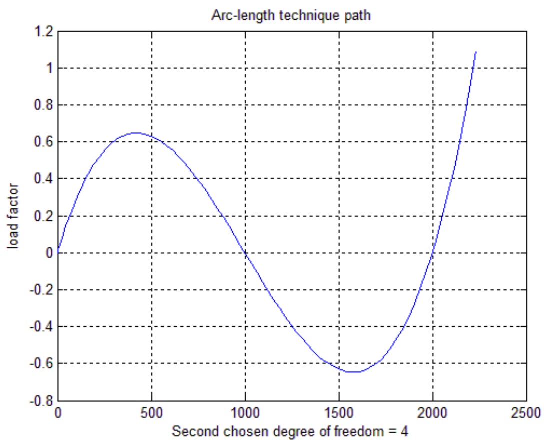

Figure 9. Load-displacement response diagram with *pr* = 2

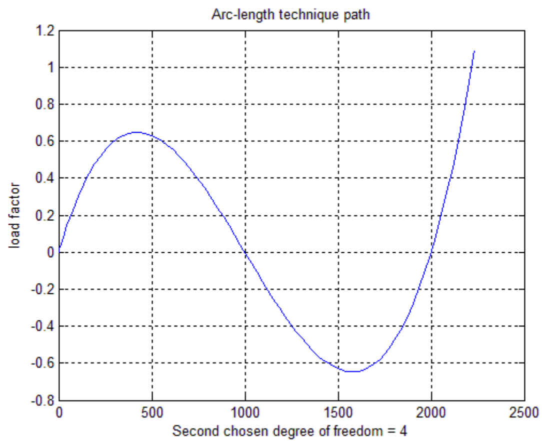

Figure 10. Load-displacement response diagram with *pr* = 3

In Table 3 and Table 4, the results of comparing the effect of changing incremental length (Δl) and comparing the effect of constraint Equation (5) and Equation (26) in implementing the arc-length method are presented. Also in this case it can be proved that the increase of incremental length (Δl) causes the points to draw curve be reduced and the new constraint equation causes the accuracy of convergence be increased. It is noticeable that the load-displacement curve will be obtained via the values of different incremental length (Δl) in Table 3 and using Equations (5) and (26) as the same Figures 9 and 10. The values of Table 4 are used for the values of the third row in Table 3.

Table 3. Comparison the effect of changing incremental length (Δl)

Initial incremental length	Minimum incremental length	Maximum incremental length	The total number of points on the path
10	5	20	222
20	10	40	111
30	15	60	74
40	20	80	56

Table 4. Comparison the effect of constraint Equations (5) and (26)

Point on the path	Number of iteration	Constraint Equation (5)	Constraint Equation (26)
1	1	-2.2737×10^{-13}	5.1699×10^{-26}
2	1	2.2737×10^{-13}	5.1699×10^{-26}
3	1	0	0
⋮	⋮	⋮	⋮
72	1	-2.2737×10^{-13}	5.1699×10^{-26}
73	1	0	0
74	1	-2.2737×10^{-13}	5.1699×10^{-26}

C. Example 3 - Six Bar Symmetric System

Until now, the behavior of the arc-length method using the criteria 1, 2 and 3 has been tested in the presence of bifurcation and limit points. With the aid of the symmetric truss structure of Figure 11, the performance of the criteria on the incidence of turning points will be examined. The equilibrium path of the truss structure of Figure 11 has eight limit points, two turning points, and at least two critical points. Figures 12 to 14 show load-displacement response diagram of the structure in Figure 11. As in the previous numerical experiment, criterion 1 fails to predict the direction of the predictor step, and the oscillation of the arc-length method around bifurcation point occurs again.

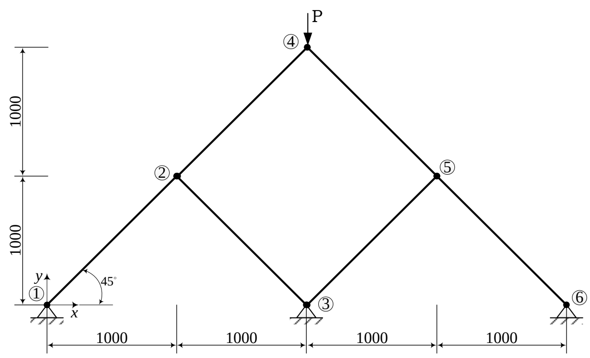

Figure 11. Six bar symmetric structure [2]

All criteria change the sign of the predictor step when passing a limit point. However, in this case, criterion 2 also fails to predict the direction of the next increment when passing a turning point. The sign of the work increment expression $(\delta d_t^0)^T q_e$ changes after passing said point, as shown in Figure 13. The arc-length control method starts to oscillate around the turning point. Finally, from the three studied criteria, the third is the only one insensible to the presence of bifurcation and turning points. Therefore, it is capable to draw complete equilibrium curve, as shown in Figure 14.

$$r(d,\lambda) = q_i(d) - \lambda q_e = 0 \quad (28)$$

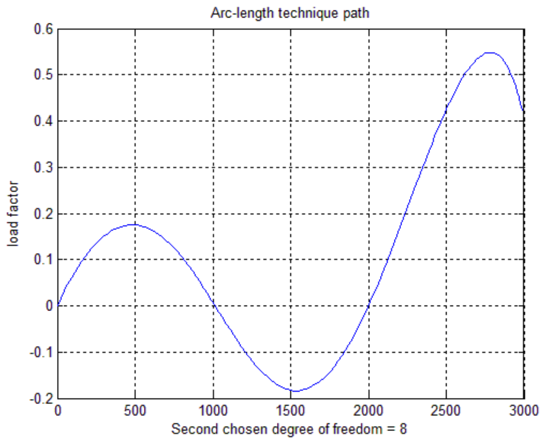

Figure 12. Load-displacement response diagram with pr = 1

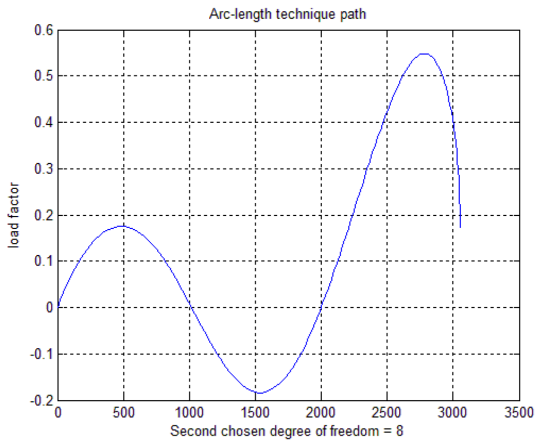

Figure 13. Load-displacement response diagram with pr = 2

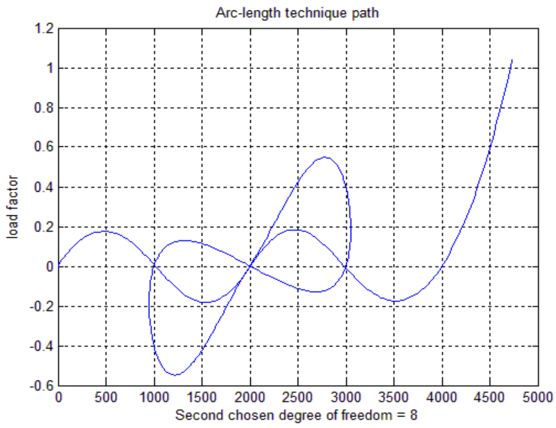

Figure 14. Load-displacement response diagram with pr = 3

Again to compare the effect of changing incremental length (Δl) and to compare the effect of constraint Equation (5) and Equation (26), Table 5 and Table 6 are presented respectively. It is noticeable that the load-displacement curve will be obtained via the values of different incremental length (Δl) in Table 5 and using Equations (5) and (26) as the same Figure 14 if the third criterion is used. The values of Table 6 are used for the values of the end row in Table 5.

Table 5. Comparison the effect of changing incremental length (Δl)

Initial incremental length	Minimum incremental length	Maximum incremental length	The total number of points on the path
10	5	20	565
20	10	40	377
30	15	60	189
40	20	70	162

Table 6. Comparison the effect of constraint Equations (5) and (26)

Point on the path	Number of iteration	Constraint Equation (5)	Constraint Equation (26)
1	1	-4.5475×10^{-13}	2.0680×10^{-25}
2	1	0	0
3	1	-1.8190×10^{-12}	0
⋮	⋮	⋮	⋮
160	1	9.0949×10^{-13}	8.2718×10^{-25}
161	1	0	7.4446×10^{-24}
162	1	-9.0949×10^{-13}	0

IX. CONCLUSIONS

By following the nonlinear equilibrium path, it is possible to understand the phenomenon of collapse or buckling or the total bearing capacity of structures. Different methods, which are incremental iterative procedure such as Newton-Raphson, are usually used to solve structures with non-linear behavior. However, those methods diverge when reach to a limit point. Arc-length method is one of the most appropriate and useful method to draw full path of the load-displacement. The arc-length method overcame the limit points when using any of the three studied criteria.

While using the determinant of the tangential stiffness matrix, to determine sign of predictor step, the arc-length method oscillated around the bifurcation points. When using the principle of the work increment, to determine the sign of the predictor step, the solution fluctuated about the turning points. However, it did not respond to bifurcations and it was able to draw the entire unstable post-bifurcation path. The criterion of the internal product of the displacement vectors was insensible to bifurcation and turning points, at least in the analyzed structures.

Therefore, when using such criterion, arc-length method was capable to draw the entire primary equilibrium path. In this method, choosing the proper size of the incremental length (Δl) can be caused to get fast convergence or divergence. In this research, the increasing of the incremental length (Δl) causes to get fast convergence but the accuracy will be reduced. To overcome this problem a new constraint equation is suggested. The new constraint equation causes the accuracy be increased.

The choice of a proper scaling parameter ψ demonstrated to have great influence on the performance of the arc-length method. In the analyzed problems, the use of $\psi = 0$ produced the best the performance of the method regarding convergence and computing time.

NOMENCLATURES

K_t: Tangent stiffness matrix
q_e: External force vector
q_i: Internal force vector

r: Residuum or out of balance vector
s: Arc length
u: Displacement field vector
δd : Iterative nodal displacement change vector
$\delta \bar{d}$: Iterative nodal displacement change
δd_t : Iterative nodal tangential displacement change
δd_t^0 : Current tangential solution
Δd: Incremental displacement nodal vector
Δd^0: Previous converged incremental displacement
Δd^1: Incremental displacement nodal vector for the predictor step
Δl: Incremental length
ψ: scaling parameter for the arc-length constraint

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