ASYMPTOTIC ANALYSIS OF FREE OSCILLATIONS FREQUENCY OF A MEDIUM CONTACTING LONGITUDINALLY REINFORCED ORTHOTROPIC CYLINDRICAL SHELL

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Abstract- In this paper, by means of the variational principle we solve a problem of free oscillation of a medium-contacting longitudinally reinforced orthotropic cylindrical shell. Based on the Ostrogradsky-Hamilton variational principle, the frequency equation of oscillations of a medium-contacting, reinforced orthotropic cylindrical shell is constructed and numerically realized. The surface loads acting by the medium on a longitudinally reinforced cylindrical shell are determined from the solutions of the system of Lame equations in displacements.

Keywords: Ribbed Shell, Variational Principle, Oscillations, Determinant, Boundary Conditions.

I. INTRODUCTION

In recent years the issues concerning the investigations of stress-strain state of medium-contacting ribbed anisotropic shells draw great attention of researchers. The papers [1, 2] were devoted to investigation of free oscillations of medium-filled isotropic cylindrical shells longitudinally reinforced and reinforced with cross system of ribs and loaded by axial compressive forces. By using the variational principle, the frequency equation of oscillations of a medium-contacting reinforced isotropic cylindrical shell is constructed and numerically realized. The free oscillations of liquid-filled ribbed isotropic cylindrical shells under axial compression was considered in the paper [3].

The shells were reinforced longitudinally, laterally and by the cross system of ribs. The similar problem considered in [3], with regard to friction between the contact surface of the shell and medium was studied in the paper [4]. In [5] a problem of forced axially symmetric oscillations of a liquid-filled isotropic cylindrical shell reinforced and loaded by axial compressive forces was studied.

II. PROBLEM STATEMENT

It is known that among the different configuration shells the ribbed cylindrical shells are the most extended ones widely used as carrying structural elements. A ribbed shell is considered as a system consisting of the actual shell and longitudinal ribs rigidly connected with it along the contact lines. It is accepted that the stress-strain state of the sheathing may be completely determined within the linear theory of elastic thin shells based on Kirchhoff-Liave hypotheses, and for calculation of ribs the theory of curvilinear Kirchhoff-Klebsch bars are applicable.

The system of coordinates were chosen so that the coordinate lines coincide with the lines of principal curvatures of the medium surface of the sheathing. It is assumed that the ribs are located along the coordinate lines, and their edges and the edges of the sheathing lie in the same coordinate plane.

The strain state of the sheathing may be determined by three components of displacements of its median surface $u, \vartheta$ and $w$. The turning angle of normal elements $\varphi_1, \varphi_2$ with respect to coordinate lines $y$ and $x$ are expressed by $w$ and $\vartheta$ with the help of dependences as:

$$\varphi_1 = -\frac{\partial w}{\partial x}, \quad \varphi_2 = -\left(\frac{\partial w}{\partial y} + \frac{\vartheta}{R}\right)$$

where $R$ is the radius of median surface of the shell.

For describing the strain state of the ribs, in addition to three components of displacements of gravity centers of their cross sections ($u_i, \vartheta_i, w_i$, respectively for the $i$th longitudinal bar) it is necessary to determine the torsion angle $\varphi_{i\varphi}$. Taking into account that according to the accepted hypotenes it holds the constancy of radial flexures along the height of cross sections, and the equality of corresponding torsion angles following from the conditions of rigid junction of ribs with the shell, we write the following relations:
\[ u_i(x) = u(x, y_i) + h_i \phi_1(x, y_i) \]
\[ \phi_i(x) = \phi(x, y_i) + h_i \phi_2(x, y_i) \]
\[ w_i(x) = w(x, y_i) \]
\[ \phi = \phi(x, y_i) \]
\[ \phi_{ki} = \phi_i(x, y_i) \]

where \( h_i = 0.5h + H_i^1 \), \( h \) is the shell thickness, \( H_i^1 \) is the distance from the axes of the \( i \)th longitudinal bar to the shell surface, \( x_i \) and \( y_i \) are the coordinates of the lines of conjunction of ribs and the shell, \( \phi_i, \phi_{ki} \) are the turning and torsion angles of cross sections of longitudinal bars.

For external actions it is assumed that the surface loads acting on the ribbed shell may be reduced to the constituents \( q_x, q_y \) and \( q_z \) applied to the median surface of the shell, and the boundary loads to longitudinal, tangential, lateral forces and bending moments applied to the edges of the shell \( T_i, S_i, Q_i, M_i \) and \( T_2, S_2, Q_2, M_2 \) on curvilinear and rectilinear edges, respectively), and to appropriate forces, bending moments and torques applied to end cross sections of the ribs \( T_i, S_i, Q_i, M_i, M_{ki} \) for longitudinal ribs).

We get differential equations of motion and natural boundary conditions for a longitudinally reinforced orthotropic cylindrical shell on the basis of the Ostrogradsky-Hamilton variational principle. For that we write beforehand the potential and kinetic energy of the system.

The potential energy of elastic deformation of a cylindrical shell has the form:

\[ \Pi_0 = \frac{hR^2}{2} \int_{x_1}^{x_2} \left[ B_{11} \left( \frac{\partial u}{\partial x} \right)^2 - 2(B_{11} + B_{12}) \frac{w}{R} \frac{\partial u}{\partial x} - \frac{B_{11}}{R} \frac{\partial w}{\partial x} \frac{\partial u}{\partial x} + \frac{B_{12}}{4} \frac{\partial w}{\partial x} \frac{\partial^2 \theta}{\partial y \partial x} \right] dx + \frac{B_{11}}{4} \frac{\partial w}{\partial x} \frac{\partial^2 \theta}{\partial y \partial x} - \frac{B_{12}}{2} \frac{w}{R} \frac{\partial \theta}{\partial y} \right] dx + \frac{B_{11}}{2} \frac{\partial \theta}{\partial y} \frac{\partial w}{\partial x} \frac{\partial \theta}{\partial y} - \frac{2(B_{11} + B_{22})}{R} \frac{\partial \theta}{\partial y} \frac{\partial w}{\partial x} \frac{\partial \theta}{\partial y} \frac{\partial w}{\partial x} \right] dx \]

where

\[ B_{11} = \frac{E_1}{1 - \nu_1 \nu_2}; \quad B_{22} = \frac{E_2}{1 - \nu_1 \nu_2}; \]
\[ B_{12} = \frac{\nu_2 E_1}{1 - \nu_1 \nu_2} = \frac{\nu_1 E_2}{1 - \nu_1 \nu_2}, \quad R \text{ is the radius of the median surface of the shell}, \quad h \text{ is the shell thickness}, \quad u, v, w \text{ are the components of displacements of the points of the medium surface of the shell}.

The expressions for the potential energy of elastic deformations of the \( i \)th longitudinal rib are as follows [7]:

\[ \Pi_i = \frac{1}{2} \int \left[ E_i F_i \left( \frac{\partial u}{\partial x} \right)^2 + E_i J_i \left( \frac{\partial^2 w}{\partial x^2} \right)^2 + \right. \]
\[ \left. + E_i J_{ki} \left( \frac{\partial^2 \theta}{\partial x \partial y} \right)^2 + 2G_i J_{ki} \frac{\partial^2 \theta}{\partial x \partial y} \frac{\partial w}{\partial x} \right] dx \]

In Equations (2) and (3) \( x_1, x_2, y_1, y_2 \) are the coordinates of curvilinear and rectilinear edges of the shell: \( F_i, J_i, J_{ki}, J_{ki} \) are area and inertia moments of the cross section of the \( i \)th longitudinal rib, respectively, with respect to the axis \( O_z \) and the axis parallel to the axis \( O_y \) and passing through the gravity center of the cross section, and also its inertia moment at torsion; \( E_i, G_i \) are elasticity and shear module of the material of the \( i \)th longitudinal bar, respectively.

The potential energy of external surface and edge loads applied to the sheathing is determined as a work performed by this loads when taking the system from one deformed state to initial underformed state and is represented in the form:

\[ A_0 = \int_{x_1}^{x_2} \left( q_x u + q_y \theta + q_z w \right) dx dy \]

\[ - \int_{y_1}^{y_2} \left( T_i u + S_i \theta + Q_i w + M_i \theta \right) dy \]

\[ + \int_{y_1}^{y_2} \left( T_2 u + S_2 \theta + Q_2 w + M_2 \theta \right) dy \]

In the similar way, the potential energies of external edge loads applied to the and faces respectively of the \( i \)th longitudinal bar, are determined by the following expressions (it accepted that only edge loads are applied to the ribs):

\[ A_i = - \int \left( T_i u + S_i \theta + Q_i w + M_i \theta \right) dy \]

\[ + \int \left( T_2 u + S_2 \theta + Q_2 w + M_2 \theta \right) dy \]

The total potential energy of the system equals the sum of potential energies of elastic deformations of the shell and longitudinal ribs, and also potential energies of all external loads:

\[ \Pi = \Pi_0 + \sum_{i=1}^{k} \Pi_i + A_0 + \sum_{i=1}^{k} A_i \]

The kinetic energy of the shell and longitudinal ribs are written in the form of:
where \( t \) is time coordinate, \( \rho_0, \rho_i \) is the density of the material from which the shell is made, the \( i \)th longitudinal bar. The kinetic energy of the longitudinally reinforced shell is:

\[
K = K_0 + \sum_{i=1}^{k} K_i
\]

The equations of the motion of the ribbed shell are obtained on the basis of Ostrogradsky-Hamilton’s principle of stationary action:

\[
\delta W = 0
\]

where \( W = \int \delta L dt \) is the Hamilton action, \( L = K - \Pi \) is the Lagrange function, \( t^r \) and \( t^m \) are the given arbitrary times.

Assuming that the shell is reinforced by an infinitely great number of ribs, by limit passage \( k_1 \to \infty \) and with respect to that variation and differentiation operations are permitted, Equation (9) may be reduced to the form:

\[
\left[ \left( a_1 + \gamma_c^{(1)} \right) \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \theta^2} \right] u + \left( 1 + a_{12} \right) \frac{\partial^2}{\partial \xi \partial \theta} - \left( a_{12} + \alpha^{(1)} \right) \frac{\partial^2}{\partial \xi^2} - \left( a_{12} + \alpha^{(2)} \right) \frac{\partial^2}{\partial \theta^2} + a_2 \frac{\partial}{\partial \theta} + a_2 w + \right.
\]

\[
\left( a_{12} - \alpha^{(1)} \right) \frac{\partial}{\partial \xi} - \left( a_{12} - \alpha^{(2)} \right) \frac{\partial}{\partial \theta} + a_2 \frac{\partial}{\partial \xi} + a_2 \frac{\partial}{\partial \theta} + a_2 \frac{\partial}{\partial \xi} + a_2 \frac{\partial}{\partial \theta} + \Delta \right]
\]

\[
\frac{\partial^2}{\partial \xi^2} + 2 \left( a_{12} + 2 \right) \frac{\partial}{\partial \xi} + \frac{\partial^2}{\partial \theta^2} + \left( a_{12} + \alpha^{(1)} \right) \frac{\partial^2}{\partial \xi^2} + 2 \left( a_{12} + 2 \right) \frac{\partial}{\partial \xi} + \frac{\partial^2}{\partial \theta^2} + \left( a_{12} + \alpha^{(2)} \right) \frac{\partial^2}{\partial \theta^2} + \Delta \right]
\]

where,

\[
a^2 = \frac{h^2}{12R^2}, \quad \bar{\rho}_c = \frac{\rho_c}{\rho_0}, \quad \Delta = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \theta^2},
\]

\[
L_i = x_2 - x_1, \quad \xi = \frac{x}{R}, \quad \gamma^{(1)}_c = \frac{E_c}{E} \left( 1 - \nu^2 \right) \gamma^{(1)}_c, \quad \eta^{(1)}_c = \frac{E_c \left( J_{w} + 4 \pi F \right)}{2 \pi R^2 h E} \left( 1 - \nu^2 \right), \quad \rho_i = 1 + \bar{\rho}_c \gamma^{(1)}_c \]
where $A, B, C$ are unknown constants, $\omega$ is a sought-for frequency. Using the solution of system (4), by displacing the points of the median surface of the shell (15) and contact conditions (13), (14), we can determine the contact pressure $q_z$. We represent this expression in the form of:

$$q_z = q_z^{(0)} C \cos n \phi \sin k x \sin \alpha t$$

where, in the case of small inertial actions of the medium on the oscillation process of the system, $q_z^{(0)}$ has the form of:

$$q_z^{(0)} = \mu_0 A^{-1} \left( 2(1-2 \nu_s) I_n (k^* + 2k' I_n' (k^*)) k^* \times \right.$$

$$\left. + \left( k^* I_n' (k^*) \right) + 2n^2 k^* \right) - 2 \left( k I_n' (k^*) - (k^* + n^2) I_n (k^*) \right) k^* \times \right.$$  

$$\times \left( 2(3-2 \nu_s) k^* I_n' (k^*) \times \right.$$

$$\left. - 2n^2 \right) + 2n \left( I_n' (k^*) - k I_n' (k^*) \right) k^* \times \right.$$  

$$\left. \left( 2(3-2 \nu_s) k^* I_n' (k^*) \times \right.$$

$$\left. - 2n^2 \right) \right)$$

When the inertial actions of the medium on the oscillation process of the system is significant, $q_z^{(0)}$ has the form of:

$$q_z^{(0)} = \frac{E}{1+\nu_s} \left[ \frac{I_n (\gamma_1)}{I_n (\gamma_1)} - \gamma_1 I_n' (\gamma_1) \left( I_n (\gamma_1) \right) + \gamma_1^* + n^2 - \frac{\nu_s}{1-2 \nu_s} - \frac{\nu_s}{\mu_0} \right]$$

$$- n^2 k^* \mu_1^* + R k^3 \gamma_1^* I_n' (\gamma_1)^* + \mu_1^* I_n (\gamma_1)^* +$$

$$\frac{k^* \gamma_1^* + I_n' (\gamma_1)^*}{\mu_1^* I_n (\gamma_1)^*} +$$

$$\frac{2n k^* \gamma_1^* \mu_1^*}{I_n (\gamma_1)^*} + \frac{2n k^* \gamma_1^* \mu_1^*}{I_n (\gamma_1)^*}.$$
where,
\[ b_1 = -\left( a_1 + \gamma_1 \right) k^2 - n^2 + \rho_1 \lambda; \] \[ b_2 = (1 + a_1) k^* n; \] \[ b_3 = -a_2 k^* + \varepsilon_1 k^*; \] \[ b_2 = -a_2 n^2; \] \[ b_3 = a_2 + a_3 \left( a_1 k^* + 2(a_1 + 2)n^2 k^2 + a_1 n^4 + \eta_1 k^4 \right) - \rho_1 \lambda + R^2 \varepsilon \alpha_1. \]

In the open form, equation (21) has the form:
\[ \lambda^3 + \alpha_1 \lambda^2 + \alpha_2 \lambda + \alpha_3 = 0 \quad (22) \]
where,
\[ \alpha_1 = \rho_1 \left( \left( \bar{b}_1 - \bar{b}_3 - a_2 n^2 \right) \right); \]
\[ \alpha_2 = \rho_1 \left[ a_1 n^2 \bar{b}_3 + \left( \varepsilon_1 k^* - a_1 \right) k^2 - \rho_1 n^2 a_2^2 - \bar{b}_3 \bar{b}_3 - a_2 n^2 \bar{b}_1 \rho_1 - \rho_1 (1 + a_1) n^2 k^2 \right]; \]
\[ \alpha_3 = \rho_1 \left[ a_2 a_3 \left( a_1 k^* + 2(a_1 + 2)n^2 k^2 + a_1 n^4 + \eta_1 k^4 \right) + + R^2 \varepsilon \alpha_1 \right]; \]

When the inertial actions of the medium on the oscillations process of the system are significant, the equation with respect to \( \lambda \) takes the form:
\[ \lambda^3 + \beta_1 \lambda^2 + \beta_2 \lambda + \beta_3 = 0 \quad (23) \]
where,
\[ \beta_1 = \left( \rho_1 \left( \rho_1 + 2 \rho_1^* \right) n \right)^{-1}; \]
\[ \beta_2 = \left( \rho_1 \left( \rho_1 + 2 \rho_1^* \right) n \right)^{-1}; \]
\[ \beta_3 = \left( \rho_1 \left( \rho_1 + 2 \rho_1^* \right) n \right)^{-1}. \]

Solving equation (22) in the domain of real numbers, we find
\[ \lambda = \sqrt[3]{\frac{q + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}{} + \frac{q}{2} - \sqrt[3]{\frac{q^2}{4} + \frac{p^3}{27} - \frac{q}{3} \frac{q^3}{27}} + \frac{p^3}{3} \frac{-q^3}{27} + \frac{3}{2} \alpha_1 \alpha_2} \quad (24) \]
\[ p = a_2 - \frac{a_2^2}{3} \left( \frac{2a_1^2}{27} - \frac{a_2^3}{3} + \alpha_3 \right) \]

Replacing in (24) \( \alpha_1, \alpha_2, \alpha_3 \) by \( \beta_1, \beta_2, \beta_3 \), respectively, we can find a real root of Equation (23). The numerical values of \( \lambda \) are found. For the problem parameters were accepted:
\[ R = 0.16 \text{ m}; \quad h = 0.00045 \text{ m}; \quad \nu_2 = 0.19; \quad \nu_1 = 0.11; \]
\[ L_1 = 0.8 \text{ m}; \quad h_0 = 0.1375 \times 10^{-3}; \quad \rho_0 = \rho_0 = 7800 \text{ kg/m}^3; \]
\[ E_c = 6.67 \times 10^9 \text{ Pa}; \quad \rho_1 = 2800 \text{ kg/m}^3; \quad m = 8; \]
\[ J_{si} = 0.8289 \times 10^{-6}; \quad \frac{J_{si}}{2\pi R^2 h} = 0.13 \times 10^{-6}; \]
\[ J_{kpi} = 0.5305 \times 10^{-6} \]

The dependence of the frequency parameter on wave formation \( n \) in the circumferential direction found from equations (22) and (23) be means of (24) are depicted in Figure 1.

![Figure 1](image1.png)

**Figure 1. Dependence of the frequency parameter on wave formation \( n \) in the circumferential direction**

![Figure 2](image2.png)

**Figure 2. Dependence of the frequency parameter on the amount of longitudinal ribs**

The values of a frequency parameter for a medium where the inertial properties are not taken into account correspond to the solid curve. The dashed line corresponds to the case when inertial properties are taken into account. Computations show that both in an isotropic shell [1-5] and for an orthotropic shell, account of inertial properties of the medium reduces to decrease of the values of the frequency parameter \( \lambda \), the growth of the parameter \( \gamma \left( \gamma = \frac{E_1}{E_2} \right) \) to increase.
Dependence of the frequency parameter on the amount of longitudinal ribs for different values of $\gamma$ is depicted in Figure 2. It is seen that with the increase in the amount of longitudinal ribs the frequency parameter of the oscillations of the constructions under investigation first increases, and then begins to decrease. The latter is connected with the fact that with increasing the amount of longitudinal ribs, the influence of their inertial properties on the oscillations process is significant. Furthermore, with the strengthening of the anisotropy property of the shell’s material, the frequency parameter of the oscillations of the studied constructions increases.

REFERENCES


BIOGRAPHIES

Fuad Seyfeddin Latifov was born in Ismayilly, Azerbaijan, in 1955. He graduated from Faculty of Mechanics-Mathematics, Azerbaijan State University, Baku, Azerbaijan in 1977. In 1983, he received his M.Sc. degree in Physics-Mathematics from Saint Petersburg State University, Saint Petersburg, Russia. In 2003, he got the Ph.D. degree in Physics-Mathematics. He is an Associate Professor and the Chief of Department of Higher Mathematics in Azerbaijan University of Architecture and Construction Baku, Azerbaijan. He has written more than 80 scientific articles and 11 monographs. He is the coauthor of an encyclopedia on mathematics.

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