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DEEP ANALYSIS OF THE MAGNETICALLY COUPLED RLC-CIRCUITS USED IN A CONVENTIONAL TESLA COIL USING SECOND ORDER DIFFERENTIAL EQUATIONS

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Abstract- In this article we are going to be looking at the theory of operation of a conventional Tesla coil, describing the physical phenomena happening in the system and having a look at the essence of the presence of the RLC coupled circuits in the Tesla coil and its consequences. We will then later analyze these RLC-circuits mathematically with the help of second order differential equations and describe the wave functions for both circuits and try to come up with technical and physical explanations for the outcomes. We are going to observe that the first and the second circuits are in great harmony with each other, since they are in resonance, delivering their own energy to the other circuit and vice versa periodically. This enthralling and captivating system has a lot of remarkable mathematics and physics behind it, worth analyzing.

Keywords: Tesla Coil, RLC-Circuits, Resonant Frequency, Transformer, Second Order Differential Equation, Spark, Generator, Primary and Secondary Circuits, Wave Functions, Quenching, Magnetic Coupling.

1. INTRODUCTION

The Tesla coil is without a doubt one of the best-known inventions of the great inventor Nikola Tesla. The Tesla coil, or as he called it "Apparatus for transmitting electrical energy" on his submitted patent in the year 1914, is a combination of several resonant circuits, theoretically only containing capacitors and inductors, which are tuned so that the circuits have the same resonant frequencies, and thereby help achieve efficient energy transformation between the circuits.

When you look at a conventional Tesla coil, you may think of it as being a huge transformer, stepping up the voltage and thereby achieving voltage gain, but the theory of operation of these two devices are totally different and utterly distinguishable. A transformer uses magnetic induction to step up the voltage, while a Tesla coil primarily uses resonant amplification to achieve the voltage gain.

It will be proved that these methods are just as effective when it comes to voltage gain through the coils, but at very high frequencies, in which we are operating, resonant amplification is just the better option, and that is why Nikola Tesla chose to incorporate the RLC-arrangement into his device instead of a conventional transformer. Although, a transformer is typically used in a Tesla coil to step up the low frequency voltage from the alternating current source to feed the primary circuit as shown in Figure 1 [5].

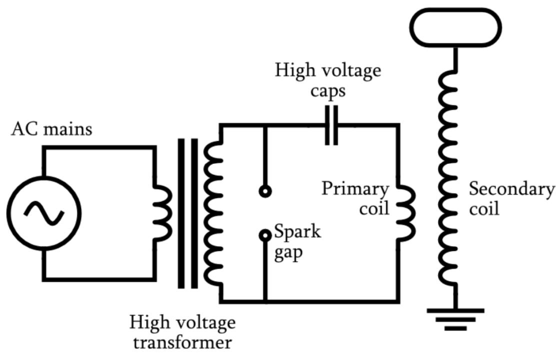


Figure 1. The circuit diagram for a conventional Tesla coil [2]

The Tesla coil goes onto producing great voltages of up to hundreds of thousands of volts and generating the enormous sparks that everybody loves to see. In this article we will be looking at the theory of operation of this device, and the mathematical equations that lay behind every physical phenomenon in this system of circuits. Let us though for now get into the theory of operation of this fascinating and unique device.

2. THEORY OF OPERATION

After stepping up the voltage with the help of the preliminary transformer we move forward to analyze the primary and the secondary circuits' characteristics. The main's frequency is typically at 50 Hz, and thus so is the voltage fed to the primary circuit by the transformer. The capacitor's capacitance is intentionally chosen so that it almost is fully charged at 1/100 of the second. The spark gap is established so that the air breakdown happens exactly at the voltage peak.

Thus, when the voltage reaches its peak value, the spark gap fires, and the lateral circuit "opens". Now we have an RLC-circuit that begins oscillating with its resonant frequency, if R is not too big, which it is not in this case. Since the secondary circuit is magnetically coupled with the primary circuit and is tuned to have the same resonant frequency as the latter, an electromotive force with the same frequency is induced in the secondary circuit. This mentioned resonant frequency is usually much higher than the main's frequency, typically between 50 kHz and 400 kHz, which is anywhere from 1,000 to 80,000 times higher than the main's frequency. As time goes on, the energy in the primary circuit is gradually transferred over to the secondary circuit. As this happens, the amplitudes of the oscillations in the primary circuit decrease, while the amplitudes of the oscillations in the secondary circuit increase. When all the energy is stored in the secondary circuit this energy is then magnetically induced in the first circuit, and the cycle repeats again and again. Since the constant of magnetic coupling is intentionally kept low at around 0.05 to 0.2, numerous oscillations are required to achieve total energy transmission from one circuit to the other.

This is the operation of the Tesla coil in a nutshell. Now, you might wonder how the secondary circuit induces an electromotive force in the primary circuit, when the voltage amplitude across the primary circuit is equal to zero, and thereby the spark gap is not excited. Is not the circuit open? The thing is that the spark gap has not had enough time to "quench" yet and the air between the spark gap is still ionized, and so the spark gap is technically still excited. This is how this process is able to keep going and repeat itself over and over again even at lower voltage amplitudes [1, 2, 6].

At the end when the voltage in the secondary circuit has reached its peak value due to resonant rise, sparks will be fired from the top load, since the air around the top load has also been ionized due to the high voltage and the masses of charges at the top load.

3. MAGNETIC INDUCTION

According to Faraday's law of induction $emf = -N\frac{d\Phi_B}{dt}$ (one of the important Maxwell equations), the rate of change of the magnetic flux $\frac{d\Phi_B}{dt}$ through a wire results in an induced electromotive force (emf) in the wire. Now since there is a proportionality between the magnitude of magnetic flux Φ_B and the magnitude of the current through the wire, indicated by i, there is also a proportionality between the rate of change of the magnetic flux $\frac{d\Phi_B}{dt}$ and the rate of change of the current $\frac{di}{dt}$ through the wire. This is also true for a coil, although an extra N-factor is needed in the equations to indicate the number of turns in the coil.

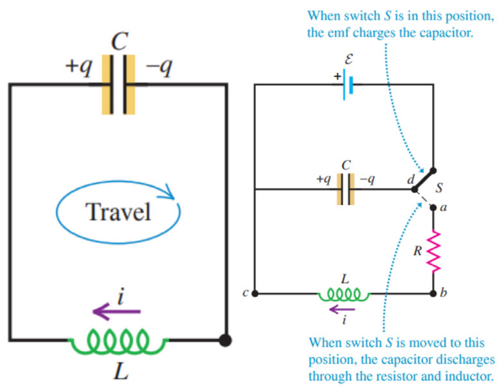
Now we define a quantity $L = \frac{N\Phi_B}{i}$, resembling the self-inductance of the coil and rewrite the equation as $N\frac{d\Phi_B}{dt} = L\frac{di}{dt}$. We will now suddenly be able to equate faraday's equation to this equation, resulting in $emf = -L\frac{di}{dt}$ for a coil's self-inductance.

The reason for the presence of this rather long reasoning is for us to understand the fact that with huge and sudden changes in the current through a coil comes drastic voltage increases or decreases through the coil, which is accomplished through self-induction.

This is one of the roles of the spark gap, providing us with a sudden discharge of the capacitor, resulting in huge and sudden current displacements, which in turn leads to huge voltage drops through the inductor due to the inductor trying to create an opposing magnetic-field relative to the current, according to Lenz's law. Now since this action is repeating, the magnetic field in the coil is an oscillating one, leading to the energy transfer from the 1.st circuit to the 2.nd circuit due to the magnetic coupling of the coils and the resonant frequency of the two circuits.

4. RLC-CIRCUITS IN PRACTICE

Let us have a quick look at how a normal RLC-circuit operates and why it oscillates at its resonant frequency.



(a) (b)

Figure 2. An LC-circuit (a), an RLC-circuit (b) [9]

We apply Kirchoff's loop rule to the circuit shown in Figure 2(a).

$$-L\frac{d^2}{dt^2}q(t) + \frac{1}{C}q(t) = 0 \tag{1}$$

Rewriting the expression gives:

$$\frac{d^2}{dt^2}q(t) - \frac{1}{LC}q(t) = 0 \tag{2}$$

Solving this differential equation for q gives:

$$q(t) = Q\cos\left(\sqrt{\frac{1}{LC}}t + \phi\right) \tag{3}$$

where, Q indicates the charge amplitude (maximal charge).

Integrating the eqution gives the expression for the current:

$$I = -\sqrt{\frac{1}{LC}} Q \sin\left(\sqrt{\frac{1}{LC}} t + \phi\right) \quad (4)$$

Now we call the term $\omega_{LC} = \sqrt{\frac{1}{LC}}$ the operating frequency of this circuit. Calculating the impedance of the circuit we get:

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \quad (5)$$

Which straight away indicates that the resonant frequency of the circuit is equall to $\omega_{res} = \sqrt{\frac{1}{LC}}$, which further indicates that the operating frequency of an LC-circuit is equal to its resonant frequency.

$$\omega_{LC} = \omega_{res} = \sqrt{\frac{1}{LC}} \quad (6)$$

In this paper when we are using the term frequency we are generally talking about the angular frequency ω and not the conventional frequency f.

Following the same steps for an RLC-circuit as Figure 2(b), we obtain the same resonant frequency $\omega_{res} = \sqrt{\frac{1}{LC}}$ but a different operating frequency equal to:

$$\omega_{RLC} = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \quad (7)$$

This indicates that our theory for the operation of Tesla coil is very accurate at low resistances, where the operating frequencies of the circuits are almost very close to their resonant frequencies. We will later analyze these circuits in full detail [8].

5. THE SECONDARY CAPACITOR

In a conventional Tesla coil the resonant frequency of the primary and the secondary circuits must be the same, in order to have maximized voltage gain. This situation is also analogous to mechanical phenomena, like the harmonic oscillation of a spring or that of atoms, when we are talking quantum mechanics. This resonant frequency is thereby equal to: $f_{res} = \frac{\omega_{res}}{2\pi} = \frac{1}{2\pi\sqrt{L_p C_p}} = \frac{1}{2\pi\sqrt{L_s C_s}}$.

The magnitude of inductances of the coils in both circuits can easily be calculated, since the inductance of such a solenoid is equivalent to $L = \frac{\mu N^2 A}{l}$ where l is the length of the solenoid, and A is the cross-sectional area of the solenoid [7].

The capacitance of the capacitor in the first circuit can easily be obtained, but the capacitance of the second circuit is usually very hard to calculate. Therefore, it is always a good idea to replace the conventional capacitor in the first circuit with a variable capacitor, so that we are able to tune the capacitor to the capacitance, where there is mutual resonance between the circuits.

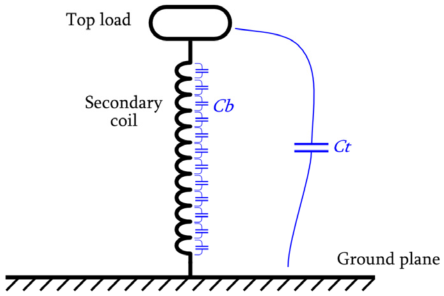


Figure 3. A sketch of some of the contributions to the diminutive capacitance of the secondary circuit [2]

The capacitance present in the second circuit can either be due to the capacitance between the top load and the ground, the relative capacitance of coil turns, or other factors. There is a clear capacitance between the top load and the ground, where the medium/dielectric in this "capacitor" is simply air. Even though the top load and ground are also connected through the secondary coil, the coil's impedance is high enough for it to not be a part of this "capacitor"

Since the wires used in the secondary coil are not ideal superconductors, the wires in the coil themselves have resistances, and since these resistances (and also the inductances) of the coil are distributed all along its length, there is a slight amount of voltage between the two adjacent turns of the coil. There are also potential differences between nonadjacent turns in the coil, but because of the longer distance these contributions are negligible. As shown in Figure 3, there are thousands of these tiny contributions, which make up a part of the secondary circuit's capacitance.

There may be all other kinds of contributions to the secondary circuit's capacitance as an example when sparks occur from the top load, there is a slight contribution to this capacitance. There is furthermore environmental factors that can play a role in this capacitance.

As it is observable in the Figure 3, all these capacitances are in parallel with each other, thus the total capacitance of the second circuit is the sum of all these minor contributions. This capacitance is extremely small, and since $L_p C_p = L_s C_s$ is the case, due to the identical resonant frequency of the circuits, the secondary circuits inductance must be tremendously large. This is usually achieved by a drastic number of windings in the secondary coil [1, 2].

6. THEORETICAL VOLTAGE GAIN

Let us try to estimate the amount of voltage gain in such a system. The following derivation is based on the conservation of energy theorem. That is there is no ohmic or nonohmic resistors or other devices that contribute to dissipation of energy in these circuits.

Since the energy is transferred from the first circuit to the second, the maximal energy in the first circuit is equal to the maximal energy in the second circuit, that is when the capacitors are fully charged in each circuit; thereby we have that: $\frac{1}{2} C_p V_p^2 = \frac{1}{2} C_s V_s^2$. The V_p indicates the rms-value of the primary voltage and the V_s indicates the rms-value of the secondary voltage. Rewriting gives $\frac{V_s}{V_p} = \sqrt{\frac{C_p}{C_s}}$, and inserting $L_p C_p = L_s C_s$ gives $gain = \frac{V_s}{V_p} = \sqrt{\frac{L_s}{L_p}}$.

Now since the magnitude of the inductance of the secondary circuit is significantly larger than the magnitude of the inductance of the primary circuit, due to the opposite capacitor ratios, the voltage gain is also tremendously high. We are talking about output voltages of up to 10^6 order of volts. Hence in a nutshell, this is how a Tesla coil is able to produce such high output voltages. You could argue that a normal transformer tends to step up the voltage with $gain = \frac{V_s}{V_p} = \frac{N_s}{N_p}$, which you might think is higher than the voltage gain achieved by resonant amplification. You are not exactly right. Because as said the inductance of a solenoid is equal to $L = \frac{\mu N^2 A}{l}$, which means that the voltage gain of a Tesla coil is also proportional to the winding ratio $gain = \frac{V_s}{V_p} \propto \frac{N_s}{N_p}$. Though since we tend to make the secondary coil longer and thinner and the primary coil shorter and thicker, the voltage gain might be very slightly less than a conventional transformer, but that's another discussion.

The actual problem with a transformer is the fact that you have to use an iron core to achieve great coupling and thereby efficient energy transformations, but when an iron core is used at very high frequencies it acts as more of a resistor rather than an assistant, and since we are working with very high voltages in a Tesla coil, resonant amplification with very low magnetic coupling constant is quite desirale and definietly the best way to go [1, 2, 3, 5].

7. EVENTUAL SOURCES OF ERROR

In this derivation we have not accounted for potential internal resistances of the circuits, which contribute to energy loss. We have used energy conservation, but we know for the matter of fact that this is practically not the case.

There are obviously various different ways, in which energy losses can occur in such systems. As an example, we can discuss the fact that since there usually are several hundred meters of wire used in a Tesla coil, there is a large resistance present in each circuit, contributing to energy loss with the rate of $P = RI^2$ if they are ohmic, and if they are non-ohmic (as they ordinarily are) the resistance is a function of temperature (and therefore so is the power *P*).

Furthermore, the impacts of the skin effect are easily perceptible on the resistance of the circuits, that is when the current flows predominantly near the surface of the wire, which occurs at substantially high frequencies and thereby increases the resistance of the entire circuit.

The spark gap itself acts kind of like a resistor, when the circuit through it is closed, dissipating energy in the form of light, heat and sound through the sparks.

The dielectric itself inside the capacitor of the primary circuit, depending on the type of dielectric used and the operation frequency, dissipates a relatively small proportion of the energy when such an alternating current is applied to its junctions.

Moreover, when operating on such high frequencies, energy loss due to emission of electromagnetic waves is inevitable. Another energy loss contribution is from the top load trying to ionize the air to be able to generate sparks.

8. QUENCHING

Quenching is the act of cooling down of the air, that is when the air is not under breakdown anymore and is not ionized. In practice we want to achieve quenching for the spark gap between the cycles, that is when the primary circuit has delivered all its energy to the secondary circuit. But the thing is, that if quenching happens at that excat moment the secondary circuit would not be able to magnetically induce the energy back to the primary circuit in the first half-cycle until the voltage reaches its peak value, thus no energy is disspiated through the spark gap in the first circuit during the first half-cycle, and thus more energy is transferred to the secondary in total without dissipation.

Quenching is achieved by having a small magnetic coupling constant and thereby weak coupling between the circuits. This gives enough time for the spark gap to quench, though the problem with having weak coupling, in other words spacing the coils further apart from each other, is that it takes many oscillations for the energy to be transferred from the primary circuit to the secondary circuit, and thereby a lot of the energy in the primary circuit is dessipated because of the spark gap. The problem with high coupling is that even though energy is transferred relatively quickly, the spark gap does not have enough time to quench, which means energy is dessipated through the spark gap in the primary circuit during the whole cycle instead, which is quite the contrary to the latter. Having an ideal coupling constant that helps us attain both of these good traits, that is being able to quench and transfer the energy fast between the circuits, at the same time is crucial for the Tesla coil's operation. This ultimately ideal magnetic coupling constant is around 0.05 to 0.2 [3].

9. THE GENERATOR

As said the primary circuit is provided with alternating current, but not always with quite a conventional one. It can actually be provided with a direct current alternating in directions as shown in Figure 4.

The switches that make this transition possible are IGBTs (short term for Insulated Gate Bipolar Transistors) as Figure 5, which can tolerate considerably high currents and voltages, which makes them ideal components to be used in power electronics systems, such as the Tesla coil itself [4].

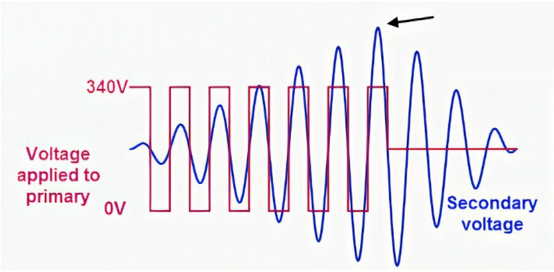


Figure 4. A possible way the gerator could provide voltage for the primary circuit [4]

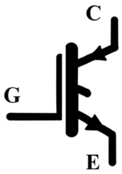


Figure 5. The schematic symbol for an IGBT

We won't go into the specifics of the IGBTs' operation, but this was just an insight into what the generator can be composed of, and what its function is.

Another very common way to excite this primary circuit in the Tesla coil with is by using a conventional alternating current generator and stepping it up to almost 10kV, and then feed this voltage to the primary circuit. A Tesla coil always tends to work with these high voltages and high frequencies, though with low intensities [1, 2].

10. COMPARING WITH THE RADIO

Some people compare the Tesla coil with the radio and even quarter-wave antennas, though these analogies are not very accurate. The greatest homogeny there is between these devices is the fact that in a radio you have to tune the variable capacitor, so that your receiver circuit has the same resonant frequency as the radio station's (the emitter's) cicuit, which is the same concept in a Tesla coil, when you are trying to receive the magnetic energy from the former circuit by having your circuit tuned at the exact same resonant frequency as the former circuit.

11. DISCOVERING THE WAVEFUNCTIONS

Now we are going to analyze an ideal Tesla coil, which contains two magnetically coupled resonant circuits as shown in Figure 6.

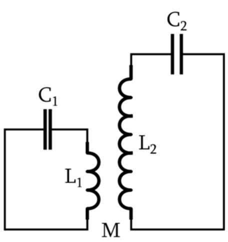


Figure 6. The schematic diagram for the magnetically coupled ideal LC-circuits of the Tesla coil [2]

Applying Kirchhoff's loop rule to both circuits, we obtain the Equations (8).

$$\begin{aligned} L_1 \frac{d^2}{dt^2} q_1(t) + M \frac{d^2}{dt^2} q_2(t) + \frac{1}{C_1} q_1(t) = 0 \\ L_2 \frac{d^2}{dt^2} q_2(t) + M \frac{d^2}{dt^2} q_1(t) + \frac{1}{C_2} q_2(t) = 0 \end{aligned} \tag{8}$$

Since the two circuits are in resonance, we know that the operating frequency of both circuits are the same, as suggested by Equation (9).

$$\omega_1 = \omega_2 = \omega_{res} \tag{9}$$

$$q_1(t) = \frac{Q}{4}\left(\cos\left(\frac{\omega_{res}}{\sqrt{1+k}} t \right) + \cos\left(\frac{\omega_{res}}{\sqrt{k-1}} t \right) \right) \tag{10}$$

$$q_2(t) = \frac{Qk}{4C_1\omega_{res}^2}\left(\cos\left(\frac{\omega_{res}}{\sqrt{1+k}} t \right) - \cos\left(\frac{\omega_{res}}{\sqrt{k-1}} t \right) \right) \tag{11}$$

where, k stands for the constant of magnetic coupling between the coils and has the condition $0 < k < 1$. Notice that the formulas can be expressed in cartesian and polar forms, though we have used Euler's formula to expand the expressions [1, 2].

We, furthermore, consider some initial conditions for the system.

$$\begin{cases} q_1(0) = Q \\ \frac{d}{dt} q_1(0) = 0 \end{cases}, \quad \begin{cases} q_2(0) = 0 \\ \frac{d}{dt} q_2(0) = 0 \end{cases} \tag{12}$$

Sketching the graphs for these functions, we obtain the results shown in Figure 7 [1, 2].

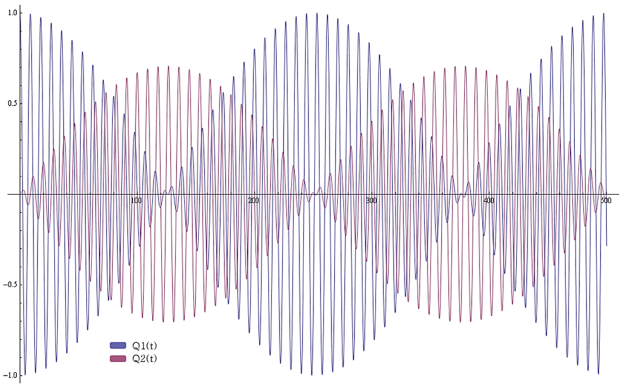


Figure 7. The graphs for the charges in both circuits relative to the time, according to Equations (10) and (11) [2]

Now it is time to look at the same circuits, but with additional resistances in series with each circuit, representing the internal resistances of the two circuits as shown in Figure 8.

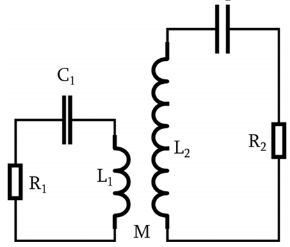


Figure 8. The schematic diagram for the magnetically coupled more realistic RLC-circuits of the Tesla coil [2]

Applying Kirchhoff's loop rule to the new circuits we obtain Equations (13).

$$L_1 \frac{d^2}{dt^2} q_1(t) + M \frac{d^2}{dt^2} q_2(t) + R_1 \frac{d}{dt} q_1(t) + \frac{1}{C_1} q_1(t) = 0$$
$$L_2 \frac{d^2}{dt^2} q_2(t) + M \frac{d^2}{dt^2} q_1(t) + R_2 \frac{d}{dt} q_2(t) + \frac{1}{C_2} q_2(t) = 0 \quad (13)$$

We again consider the two operating frequencies to be equal, according to Equation (9). There is actually no analytical solution for the differential equations given by Equation (13). These equations must be resolved numerically. Implementing this system in a computing software using the same initial conditions as the precious system, given by Equation (12), we obtain the results shown in Figure 9 [1, 2].

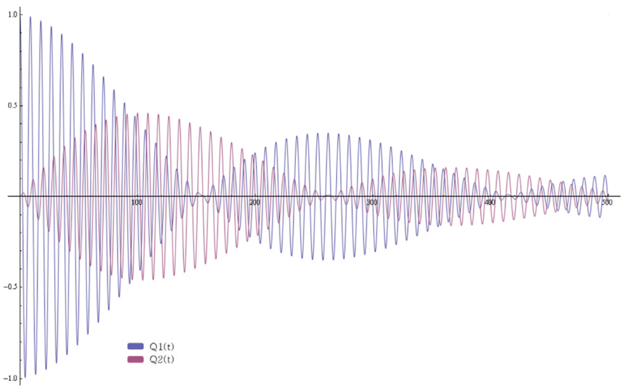


Figure 9. The graphs for the charges in both circuits relative to the time, acquired by solving Equation (13) numerically [2]

12. CONCLUSION

The graphs shown in Figures 7 and 9 correspond well to our aforementioned theories. The energies get transferred periodically from one of the circuits to the other and vice versa. There are shifts in the frequencies of the two induced voltages in Figure 9, since there are resistances present in both circuits that manipulate the resonant frequency and thereby the operating frequency. The shifts are not homogenous due to the fact that the resistance of the secondary circuit is considerably larger than the resistance of the primary circuit, which has been accounted for during our calculations [1, 2].

Now we understand why the Tesla coil is so much different than a conventional transformer and why it is so efficient when it comes to energy transformation at high frequencies compared to an ordinary transformer.

This astonishing invention by Nikola Tesla is only one of the few outstanding inventions of his. Tesla goes on to inventing the first radio (even before Marconi), the first ex-ray and the first AC motor in the history of mankind. These are still only a few examples of his great work. Tesla is the inventor of the future.

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BIOGRAPHY

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